1

Introduction

- In this topic, we will
  - Focus on 1st-order initial value problems (IVPs)
  - Observe how solving such solutions is equivalent to performing integration
  - List the methods we will use to approximate solutions
  - Observe that these solutions parallel algorithms we've already seen
  - Show the meaning of the ordinary differential equation and the initial conditions visually

2

Initial-value problems

- A first-order initial-value problem (IVP) is any such problem that can be written as:

$$y^{(1)}(t) = f\left(t, y(t)\right)$$
$$y(t_0) = y_0$$

  - In calculus, techniques used in solving higher-order IVP, boundary-value problems, and partial differential equations often differ
  - The techniques we will use to approximate solutions, however, can be generalized

Our approach

- We could immediately jump to explaining the Dormand-Prince method…
  - After all, this algorithm was specified in 1980
    - An algorithm likely defined within the lifetime of your parents
- Unfortunately, you would not understand the issues at hand
  - Instead, we will build up with simpler solutions

Integration

- It's also useful to remember that this is integration:

$$y^{(1)}(t) = f\left(t, y(t)\right)$$

$$\int_{t_0}^{t} y^{(1)}(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} f\left(t, y(t)\right)\mathrm{d}\tau$$

$$\left(y(\tau)\right)\Big|_{t_0}^{t} = \int_{t_0}^{t} f\left(t, y(t)\right)\mathrm{d}\tau$$

$$y(t) - y(t_0) = \int_{t_0}^{t} f\left(t, y(t)\right)\mathrm{d}\tau$$

$$y(t) = y(t_0) + \int_{t_0}^{t} f\left(t, y(t)\right)\mathrm{d}\tau$$

Initial condition

Integration

- Recall that in solving an initial-value problem, we are integrating
  - Consequently, the techniques are very similar

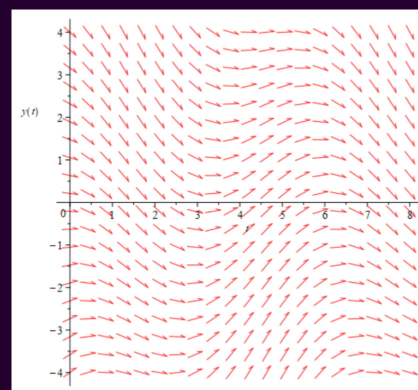
IVP algorithm	Integration algorithm
Euler's method	Riemann sums
Heun's method	Trapezoidal rule
4th-order Runge-Kutta method	Simpson's rule
Adaptive Euler-Heun method	n/a
Adaptive Dormand-Prince method	n/a
Backward-Euler method	n/a

Visual interpretation

- To understand how these techniques work, it is useful to understand what the ordinary differential equation means visually
  - If $y^{(1)}(t) = f\left(t, y(t)\right)$, then the slope of $y(t)$ for any t must equal the function in question

$$y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$$

$f\left(t, y(t)\right)$

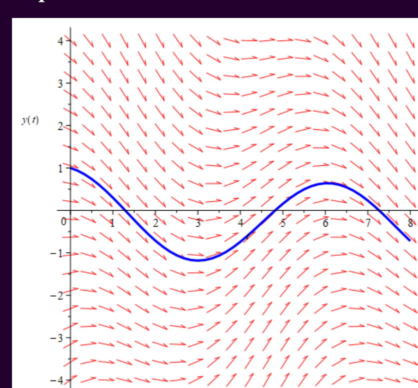


Approximating solutions to 1st-order initial-value problems

Visual interpretation

- For example, if $y(0) = 1$, there is only one function that satisfies this condition and the differential equation

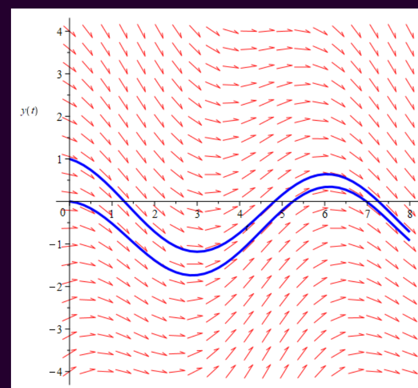
$$y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$$



Visual interpretation

- Similarly, if $y(0) = 0$, again, there is only one function that satisfies both this condition and the differential equation

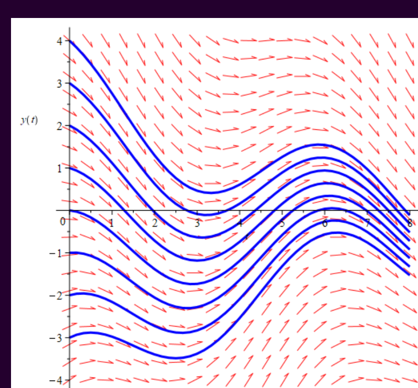
$$y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$$



Visual interpretation

- Indeed, any initial condition you specify results in a different solution
  - Our goal will be to approximate these solutions

$$y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$$



Approximating solutions to 1st-order initial-value problems

Visual interpretation

- Fortunately, we will use the differential equation:

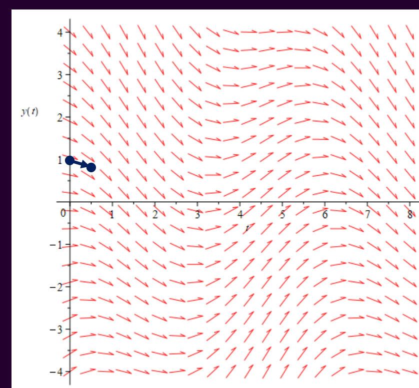
$$y^{(1)}(t) = -0.2y(t) - \sin(t) - 0.1$$

  - Suppose we have an initial condition $y(0) = 1$
  - The slope here

$$y^{(1)}(0) = -0.2 \cdot 1 - \sin(0) - 0.1 = -0.3$$

  - From Taylor series, you know that

$$y(0+h) \approx y(0) + hy^{(1)}(0)$$



11

11

Approximating solutions to 1st-order initial-value problems

Summary

- Following this topic, you now
  - Have a better understanding of an initial-value problem
  - Are aware of how approximating such solutions are equivalent to performing integrations
  - Understand how the differential equation gives the slope of solutions
  - Understand how the initial condition specifies one specific solution

12

12

References

[1] https://en.wikipedia.org/wiki/Initial_value_problem

13

13

Approximating solutions to 1st-order initial-value problems

Acknowledgments

None so far.

14

14

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

Disclaimer

These slides are provided for the ECE 204 *Numerical methods* course taught at the University of Waterloo. The material in it reflects the author’s best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.